Stability Analysis of Interconnected Deformable Bodies in a Topological Tree

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The exact nonlinear rotational equations of a system of interconnected deformable bodies are derived from the Alembert virtual work principle. The Lagrange equations of the system are also written as they lead to a rather simple stability analysis. Equilibrium conditions, permitting the equilibrium synthesis, are derived. The method is appropriate for simple numerical structure design and stability investigation.

Introduction

THE paper deals with the rotational dynamics of nonrigid mechanical systems. Such a system can be modeled as a continuum or as a set of interconnected rigid bodies. These two approaches have been developed in the literature during the last decade. This paper could be related to the so-called hybrid coordinates approach introduced by Likins[1] as both discrete and distributed coordinates will be considered.

In the traditional hybrid formalism discrete coordinates describe the motion of the rigid parts of the system and distributed coordinates permit the description of the motion of particular deformable parts. Here discrete coordinates will describe the relative orientation of distinct deformable parts.

The final goal of the paper is to obtain stability conditions for a given equilibrium state. As a by product of the analysis a set of equilibrium conditions is obtained.

In a previous paper[2] the authors determined rotational equilibrium and stability conditions for a set of point-connected rigid gyrostats in a topological tree configuration. Their equations of motion were derived from the Roberson-Wittenburg formalism[3] and expressed in a mean body frame.

The problem considered here is similar; however, the formalism allows for interconnections with up to six degrees of freedom and the various bodies may undergo (visco-) elastic deformations.

The equations of a similar system were obtained by Roberson[4] and our purpose will be to present them in a form suitable for stability analysis and simulation.

Little attention was paid to the choice of a reference frame as it appears that this choice does not substantially modify the analysis.[5] We will just state that the body reference frame is centered at the center of mass of the whole system and that its relative orientation with respect to the structure is determined by the choice of variables.

The authors were confronted with the difficult choice of a formalism for the derivation of the equations. They abandoned the previous approaches to the problem[6] to return to the more fundamental Alembert's virtual work principle (applied to the whole system).

This permits to derive the exact nonlinear equations of motion (allowing for large deformations in the structure). Such equations are realistic and can be used as such for dynamic simulation and equilibrium investigation. Here also, the authors' intention was to avoid the lengthy traditional procedure and suggest that the equilibrium synthesis be carried out by adjusting a certain number of free parameters in order to satisfy the equilibrium conditions.

The paper is restricted to topological tree configurations. When closed-loops are included in the system, the basic analysis must be substantially modified and this problem can be considered separately. The analysis permits to obtain the stability conditions in a rather simple analytical form.

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Description of the System and Choice of Variables

The basic notations of the Roberson-Wittenburg formalism will be used. In particular the interconnected bodies will be put into correspondance with a topological graph. The bodies will be considered as vertices and the connections as arcs. The choice of arc orientation, as well as the arcs' and vertices' labelings, will be kept optional. Furthermore, an arbitrary body will be chosen as reference body and referred to as the rth body. The important notions of augmented bodies and barycentric vectors will have to be adapted to the actual problem and the auxiliary but useful notion of extended body will be introduced.

The physical joints will be modeled by massless translational systems represented by $\mathbf{z}^a$-vectors joining the body attachment points and oriented along the corresponding arcs and by the rotational joints a located at the tip of these $\mathbf{z}^a$-vectors. When translation is not permitted, $\mathbf{z}^a$ vanishes and the rotational joints a are located at the corresponding connection points.

An extended body i is composed of the ith body and the translational systems whose origin lies on it (Fig. 1). The ith augmented body then consists of the ith extended body with mass m^i and point masses m^{ia}, located at the joints a, and equal to the total mass of the subgraphs not containing body i formed when arc a is cut.

The barycenter of the ith body B^i is thus the mass center of the ith augmented body. In body i, the vector joining B^i to the joint a (tip of $\mathbf{z}^a$) lying on the minimum path to body j, $[i-j]$, (arc d, $[i-d]$) is denoted by $\mathbf{b}^{ij}$ or $\mathbf{b}^{ia(j)}$ ($\mathbf{b}^{id}$ or $\mathbf{b}^{ia(d)}$); the vector $\mathbf{b}^{ii}$ (sometimes denoted $\mathbf{b}^i$, for convenience) will be the vector joining the barycenter to the mass center of the body.

From the definition of barycentric vectors, it turns out that

$$\sum_j m^j \mathbf{b}^{ij} = \sum_a S^{ia} m_*{}^{ia} \mathbf{b}^{ia} + m^i \mathbf{b}^i = 0 \tag{1}$$

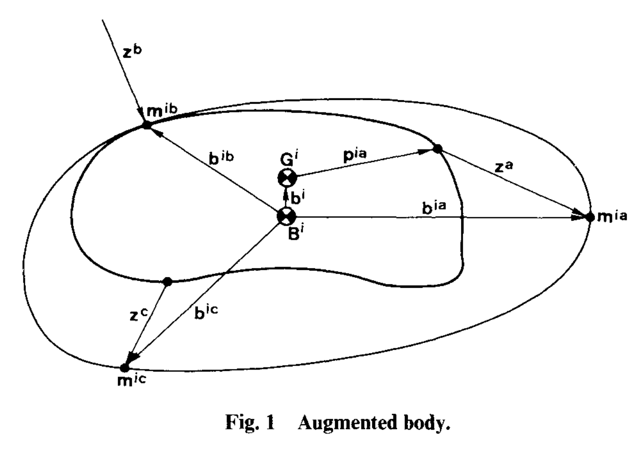

Fig. 1 Augmented body.

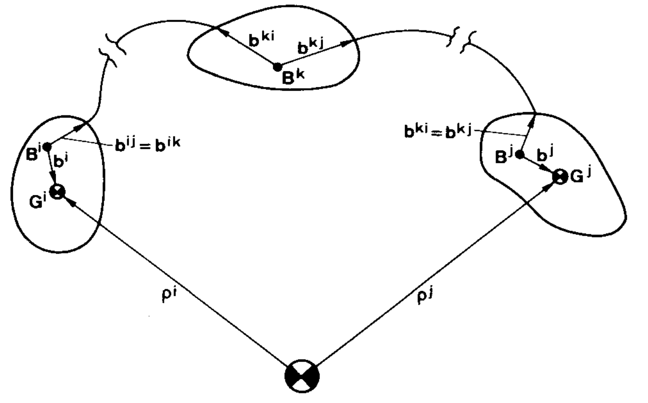

Fig. 2 Barycentric vectors.

where S^{ia} are elements of the graph incidence matrix defined as

$$S^{ia} = \begin{cases} 1 & \text{if arc } a \text{ starts from body } i \\ -1 & \text{if arc } a \text{ terminates on body } i \\ 0 & \text{otherwise} \end{cases}$$

$m_*{}^{ia}$ are graph parameters defined as

$$m_*{}^{ia} = \begin{cases} -m^{ia} & \text{when the arc } a \text{ is oriented towards the body } i \\ m^{ia} & \text{when the arc } a \text{ is oriented in the opposite direction} \end{cases}$$

If $\mathbf{p}^{ia}$ is the position vector of the attachment point of the translational system of arc a in body i, with respect to G^i, $\mathbf{b}^{ia}$ can be written

$$\mathbf{b}^{ia} = \mathbf{b}^i + \mathbf{p}^{ia} + \bar{S}^{ia}\mathbf{z}^a \tag{2}$$

where the elements of the modified incidence matrix $\bar{S}$ are

$$\bar{S}^{ia} = \begin{cases} 1 & \text{when } S^{ia} = 1 \\ 0 & \text{otherwise} \end{cases}$$

The total mass of the system, m, can be written

$$m = \sum_j m^j = m^i + \sum_a S^{ia} m_*{}^{ia} = \sum_j S^{ja} m_*{}^{ja} \tag{3}$$

and using Eqs. (1) and (2)

$$\mathbf{b}^i = -\sum_c S^{ic}(m_*{}^{ic}/m)(\mathbf{p}^{ic} + \bar{S}^{ic}\mathbf{z}^c) \tag{4}$$

Taking into account the fact that $m_*{}^{ia} = m_*{}^{ja}$ when i and j belong to the same subgraph with respect to a and splitting up in Eq. (4) the summation on c into summations on $c = a$ and $c \neq a$, we obtain by Eqs. (2) and (4)

$$\mathbf{b}^{ia(j)} = -\sum_c S^{jc}(m_*{}^{jc}/m)(\mathbf{p}^{jc} + \bar{S}^{jc}\mathbf{z}^c) \tag{5}$$

The vector ρ^j joining the mass center G of the whole system to the mass center G^j of the body j will be fruitfully expressed in terms of the barycentric vectors. We will now prove from simple geometrical considerations the relation

$$\rho^j = \sum_k \mathbf{b}^{kj}$$

obtained by Roberson in Ref. 3.

From Fig. 2, we have

$$\rho^i - \rho^j = \sum_{k \in [i-j]} (\mathbf{b}^{ki} - \mathbf{b}^{kj}) = \sum_k (\mathbf{b}^{ki} - \mathbf{b}^{kj})$$

for if $k \notin [i-j]$, $\mathbf{b}^{ki} = \mathbf{b}^{kj}$. We then multiply the two sides of the above relation by m^i, and sum on i. The result follows from Eqs. (1) and (3) and the trivial relationship

$$\sum_i m^i \rho^i = 0$$

We will also need the following properties:

$$\sum_i m^i m_*{}^{ia} = 0 \tag{6}$$

$$\sum_i m^i (m_*{}^{ia}/m)\mathbf{b}^{ki} = -m_*{}^{ka}\mathbf{b}^{ka} \tag{7}$$

If 1 and 2 are the bodies adjacent to a, one can write

$$\sum_i m^i m_*{}^{ia} = S^{2a} m_*{}^{2a} m_*{}^{1a} + S^{1a} m_*{}^{1a} m_*{}^{2a} = 0$$

by definition of $m_*{}^{ia}$.

If the notation $a < i$ means that the arc a lies on the minimum path between i and r (i must then be different from r and the notation $a \nless i$ means that a does not lie on this direct path), then the two subgraphs obtained by cutting the arc a include the bodies with indices i such that $a < i$ and $a \nless i$ respectively. If $k: a < k$, the left side of Eq. (7) can be written

$$\sum_i m^i (m_*{}^{ia}/m)\mathbf{b}^{ki} = \sum_{i:a<i} (m^i \mathbf{b}^{ki}) m_*{}^{ka}/m + \sum_{i:a \nless i} (m^i m_*{}^{ia}/m)\mathbf{b}^{ka}$$

and the relation (7) follows from the use of Eq. (1) in the first summation and of Eq. (6) in the second summation.

An orthonormal frame $\{\hat{\mathbf{X}}_\alpha{}^i\}$ will be associated with each body i. Its origin will always remain at the center of mass G^i and it will be constrained to follow the body's rigid motion.

The position vector of an element of mass dm^i with respect to G^i will be denoted $\mathbf{p}^i$ during deformation and $\mathbf{x}^i$ in the undeformed (equilibrium) configuration.

Furthermore, a general body frame $\{\hat{\mathbf{X}}_\alpha\}$ will be defined. The origin of this frame coincides with the center of mass of the whole body and it will be assumed that $\{\hat{\mathbf{X}}_\alpha\}$ is parallel to $\{\hat{\mathbf{X}}_\alpha{}^r\}$.

Any vector $\mathbf{v}^i$ will then be expressed as

$$\mathbf{v}^i = [\hat{\mathbf{X}}_\alpha{}^i]^T \underline{v}^i = [\hat{\mathbf{X}}_\alpha]^T v^i$$

where

$$[\hat{\mathbf{X}}_\alpha] = [\hat{\mathbf{X}}_1 \hat{\mathbf{X}}_2 \hat{\mathbf{X}}_3]^T; \quad [\hat{\mathbf{X}}_\alpha{}^i] = [\hat{\mathbf{X}}_1{}^i \hat{\mathbf{X}}_2{}^i \hat{\mathbf{X}}_3{}^i]^T$$

$$\underline{v}^i = [\underline{v}_1{}^i \underline{v}_2{}^i \underline{v}_3{}^i]^T; \quad v^i = [v_1{}^i v_2{}^i v_3{}^i]^T$$

$\underline{v}_\alpha{}^i$ and $v_\alpha{}^i$ being the α components of $\mathbf{v}$ in the $\{\hat{\mathbf{X}}_\alpha{}^i\}$- and $\{\hat{\mathbf{X}}_\alpha\}$-frame, respectively. A vector with two indices will always be expressed in the frame corresponding to the first index, i, e.g., $\mathbf{v}^{ij} = [\hat{\mathbf{X}}_\alpha{}^i]\underline{v}^{ij}$.

The transformation matrix between two frames will be given by

$$[\hat{\mathbf{X}}_\alpha{}^i] = A^{ij}[\hat{\mathbf{X}}_\alpha{}^j]$$

and then

$$v^i = A^{ri}\underline{v}^i$$

Similarly any tensor $\mathbf{T}^i$ will be written

$$\mathbf{T}^i = [\hat{\mathbf{X}}_\alpha{}^i]^T \underline{T}^i [\hat{\mathbf{X}}_\alpha{}^i] = [\hat{\mathbf{X}}_\alpha]^T T^i [\hat{\mathbf{X}}_\alpha]$$

with

$$T^i = A^{ri}\underline{T}^i A^{ir}$$

Auxiliary reference frames will be fixed to each part of the rotational joints a and the relative angular rate of the "tip" frame with respect to the "origin" frame, $\{\hat{\mathbf{X}}_\alpha{}^a\}$, will be denoted $\mathbf{\Omega}^a$. These frames are fixed in their respective bodies relative to the attachment point. The $\{\hat{\mathbf{X}}_\alpha{}^a\}$-frame belongs to the body at the origin of $\mathbf{z}^a$, and its relative angular velocity with respect to the corresponding body frame $\{\hat{\mathbf{X}}_\alpha{}^k\}$ will be denoted $\mathbf{\Omega}^{ka}$.

The angular velocity of the $\{\hat{\mathbf{X}}_\alpha{}^i\}$-frame with respect to inertial space will be denoted ω^i.

The elements of the graph matrix T are defined as follows:

$$T^{ai} = \begin{cases} 1 & \text{if } a < i \text{ and arc } a \text{ is pointing towards } r, \\ -1 & \text{if } a < i \text{ and } a \text{ is not pointing towards } r, \\ 0 & \text{otherwise.} \end{cases}$$

It should be noted that this matrix is a left pseudo-inverse of the incidence matrix S.

The angular velocity of body i is thus related to the angular velocity of the reference body $\omega^r = \omega$ by the relation

$$\omega^i = \omega - \sum_a T^{ai}\mathbf{\Omega}^a - \sum_a \sum_k T^{ai} S^{ak} \mathbf{\Omega}^{ka} \tag{9}$$

The deformation of the structure will be described by deformation variables, $\beta_k{}^i$ ($k = 1, \ldots, n^i$, n^i being the number of internal degrees of freedom in body i). The displacement of an element of mass dm^i, i.e., $\mathbf{u}^i = \mathbf{p}^i - \mathbf{x}^i$ will be a function of the position vector and of the deformation variables β^i

$$\mathbf{u}^i = \mathbf{u}^i(\mathbf{x}^i, \beta^i)$$

where $\beta^i = [\beta^i, \ldots, \beta_{n^i}{}^i]^T$.

As an example, in a finite element model the deformation variables could be the displacements with respect to the equilibrium state, and the angular rotations of characteristic material points, the displacement field being described by interpolation functions. In a modal approach, the internal variables would be the amplitudes of the modes.

The corresponding deformation, local rotation and stress fields will be respectively written as

$$\varepsilon^i = \varepsilon^i(\mathbf{x}^i, \beta^i), \quad \alpha^i = \alpha^i(\mathbf{x}^i, \beta^i), \quad \sigma^i = \sigma^i(\varepsilon^i)$$

The forces and torques applied through the rotational joint a by the body lying at the tip of arc a, on the body lying at its origin will be written $\mathbf{F}^a$ and $\mathbf{L}^a$, respectively. They will be expressed in $\{\hat{\mathbf{X}}_\alpha{}^a\}$ as $\mathbf{F}^a = [\hat{\mathbf{X}}_\alpha{}^a]^T \underline{F}^a$ and $\mathbf{L}^a = [\hat{\mathbf{X}}_\alpha{}^a]^T \underline{L}^a$.

The effect of rigid constant speed rotors included in the system will be represented by constant norm internal angular momentum vectors $\mathbf{h}^{is}$ aligned with their rotor axes. The relative angular velocity of these vectors with respect to the $\{\hat{\mathbf{X}}_\alpha{}^i\}$-frame will be denoted $\mathbf{\Omega}^{is}$ and, as $\mathbf{\Omega}^{ia}$, will be functions of the deformation variables.

Equation of Motion

The equation of motion will be derived from the Alembert's virtual work principle.

The virtual work done by external forces and inertial forces during a virtual displacement compatible with the constraints is equal to the corresponding virtual change in potential energy of deformation.

The virtual displacement of a position vector $\mathbf{v}^i$ expressed in the $\{\hat{\mathbf{X}}_\alpha{}^i\}$-frame can be written

$$\delta\mathbf{v}^i = \delta[\hat{\mathbf{X}}_\alpha{}^i]^T \underline{v}^i + [\hat{\mathbf{X}}_\alpha{}^i]^T \delta\underline{v}^i$$

or

$$\delta\mathbf{v}^i = \boldsymbol{\delta\psi}^i \times \mathbf{v}^i + [\hat{\mathbf{X}}_\alpha{}^i]^T \delta\underline{v}^i$$

where $\boldsymbol{\delta\psi}^i$ is the pseudo-vector of virtual rotation of the $\{\hat{\mathbf{X}}_\alpha{}^i\}$-frame with respect to inertial space (the components of this vector are then quasi-coordinates).

The virtual rotation vector of the joint a will be written as

$$\boldsymbol{\delta\gamma}^a = [\hat{\mathbf{X}}_\alpha{}^a]^T \underline{\delta\gamma}^a$$

The virtual rotation vector of the $\{\hat{\mathbf{X}}_\alpha{}^a\}$-frame with respect to $\{\hat{\mathbf{X}}_\alpha{}^i\}$ will be denoted

$$\boldsymbol{\delta\gamma}^{ia} = [\hat{\mathbf{X}}_\alpha{}^i]^T \underline{\delta\gamma}^{ia}$$

Similarly, the virtual rotation vector of the rotor angular momentum will be denoted

$$\boldsymbol{\delta\gamma}^{is} = [\hat{\mathbf{X}}_\alpha{}^i]^T \underline{\delta\gamma}^{is}$$

The virtual rotations $\underline{\delta\gamma}^{ia}$ and $\underline{\delta\gamma}^{is}$ are related to the rotational field and the virtual changes of the deformation variables. These relations will be assumed to have the form

$$\underline{\delta\gamma}^{ia} = \underline{\Phi}^{ia}(\partial\alpha^{ia}/\partial\beta^i)\,\delta\beta^i$$

where the components of α^{ia} describe the orientation of the attachment frame (in body i) of joint a with respect to the $\{\hat{\mathbf{X}}_\alpha{}^i\}$-frame; the matrix $\underline{\Phi}^{ia}$ depends on the choice of the rotation variables and is defined by the relation

$$\underline{\Omega}^{ia} = \underline{\Phi}^{ia}\dot{\alpha}^{ia} = \underline{\Phi}^{ia}(\partial\alpha^{ia}/\partial\beta^i)\dot{\beta}^i$$

From the virtual work principle, we obtain the following equation

$$\sum_i \int_i \sigma^{iT}\,\delta\varepsilon^i\,dv + \sum_a \mathbf{F}^a \cdot [\hat{\mathbf{X}}_\alpha{}^a]^T \delta\underline{z}^a + \sum_a \mathbf{L}^a \cdot \boldsymbol{\delta\gamma}^a +$$
$$\sum_i \int_i \{[-\mathbf{f}^i + (\ddot{\mathbf{R}} + \ddot{\boldsymbol{\rho}}^i + \ddot{\mathbf{p}}^i)] \cdot \delta(\mathbf{R} + \boldsymbol{\rho}^i + \mathbf{p}^i)\}\,dm +$$
$$\sum_{i,s} \dot{\mathbf{h}}^{is} \cdot (\boldsymbol{\delta\psi}^i + \boldsymbol{\delta\gamma}^{is}) = 0 \qquad (10)$$

where $\mathbf{f}^i$ is the external force per unit mass and $\mathbf{R}$ is the position vector of the general center of mass (with respect to inertial space).

The virtual changes will be expressed in terms of the virtual changes of independent variables consistent with the constraints imposed on the system. The number of these variables will then be equal to the number of degrees of freedom.

We will choose the following independent variables δR, $\delta\psi$, $\underline{\delta\gamma}^a$, $\delta\underline{z}^a$, $\delta\beta^i$, where δR and $\delta\psi$ correspond to the six rigid degrees of freedom (translation and rotation) of the general system and $\delta\beta^i$ is an n_i vector. The effective dimension of $\delta\underline{z}^a$ and $\underline{\delta\gamma}^a$ depends on the number of degrees of freedom in the joint a. For instance, for a hinge two of the components of the matrix $\underline{\delta\gamma}^a$ will be equal to zero if $\boldsymbol{\delta\gamma}^a$ is aligned with the axis.

From the abovementioned definitions, one can write

$$\delta\varepsilon^i = (\partial\varepsilon^i/\partial\beta^i)\,\delta\beta^i$$
$$\delta\underline{u}^i = (\partial\underline{u}^i/\partial\beta^i)\,\delta\beta^i$$
$$\delta\mathbf{p}^i = \boldsymbol{\delta\psi}^i \times \mathbf{p}^i + [\hat{\mathbf{X}}_\alpha{}^i]^T \delta\underline{u}^i$$
$$\delta\mathbf{z}^a = \sum_i \bar{S}^{ia}(\boldsymbol{\delta\psi}^i + \boldsymbol{\delta\gamma}^{ia}) \times \mathbf{z}^a + [\hat{\mathbf{X}}_\alpha{}^a]^T \delta\underline{z}^a$$

the corresponding $\boldsymbol{\delta\psi}^i$ being related to the independent virtual changes by a relation similar to Eq. (9), i.e.,

$$\boldsymbol{\delta\psi}^i = \boldsymbol{\delta\psi} - \sum_a T^{ai}\,\boldsymbol{\delta\gamma}^a - \sum_{a,k} T^{ai}S^{ak}\,\boldsymbol{\delta\gamma}^{ka}$$

The relation (10) must be satisfied for any virtual displacement and as the δ-variables are assumed independent their coefficients must vanish. We then obtain a set of n differential equations in the n independent variables, the equations of motion. Moreover, these equations are obtained in vectorial form.

The equations of translation and rotation of the complete system are obtained by equating to zero the coefficients of $\delta\mathbf{R}$ and $\boldsymbol{\delta\psi}$, or

$$m\ddot{\mathbf{R}} - \sum_i \mathbf{F}^i = 0 \qquad (11)$$

and

$$\sum_i \int_i \mathbf{p}^i \times \ddot{\mathbf{p}}^i\,dm + \sum_{i,j,k} m^j\mathbf{b}^{ij} \times \ddot{\mathbf{b}}^{kj} + \sum_{i,s} \dot{\mathbf{h}}^{is} -$$
$$\sum_i \int \mathbf{p}^i \times \mathbf{f}^i - \sum_{i,j} \mathbf{b}^{ij} \times \mathbf{F}^j = 0 \qquad (12)$$

The first equation is the general translational equation which in most cases is not coupled with the other equations and it will not be considered in the sequel. Equation (12) is none other than the Euler-Liouville-Resal equation of the whole system. It can be obtained from $\dot{\mathbf{H}} = \mathbf{L}$ where $\mathbf{H}$ is the total angular momentum with respect to G and $\mathbf{L}$ is the resultant of external torques.

The coefficients of $\delta\underline{z}^d$ provide the system

$$\sum_i m_*{}^{id}(\mathbf{b}^{id} + \mathbf{F}^i/m) + \mathbf{F}^d = 0 \qquad (13)$$

The coefficients of $\underline{\delta\gamma}^b$ provide the rotational equations of the joints

$$-\sum_i T^{bi} \int_i \mathbf{p}^i \times \ddot{\mathbf{p}}^i\,dm - \sum_{i,j,k} T^{bi}m^j\mathbf{b}^{ij} \times \ddot{\mathbf{b}}^{kj} - \sum_{i,s} T^{bi}\dot{\mathbf{h}}^{is} +$$
$$\sum_i T^{bi} \int_i \mathbf{p}^i \times \mathbf{f}^i\,dm + \sum_{i,j} T^{bi}\mathbf{b}^{ij} \times \mathbf{F}^j + \mathbf{L}^b = 0 \qquad (14)$$

In the case of a hinge joint, or a translational system with less than three degrees of freedom, the only components of these equations that must be considered are the ones corresponding to the degrees of freedom.

Finally, the coefficients of $\delta\beta^i$ provide the deformation equations

$$\int_i \left(\frac{\partial\varepsilon^i}{\partial\beta^i}\right)^T \sigma^i\,dv - \int_i \left(\frac{\partial u^i}{\partial\beta^i}\right)^T [\hat{\mathbf{X}}^i] \cdot (\mathbf{f}^i - \ddot{\mathbf{p}}^i)\,dm -$$
$$\sum_a S^{ia}\left(\frac{\partial u^{ia}}{\partial\beta^i}\right)^T [\hat{\mathbf{X}}^i] \cdot \mathbf{F}^a -$$
$$\sum_a S^{ia}\left(\frac{\partial\alpha^{ia}}{\partial\beta^i}\right)^T \underline{\Phi}^{iaT}[\hat{\mathbf{X}}^i] \cdot (\mathbf{L}^a + \bar{S}^{ia}\mathbf{z}^a \times \mathbf{F}^a) +$$
$$\sum_s \left(\frac{\partial\alpha^{is}}{\partial\beta^i}\right) \underline{\Phi}^{isT}[\hat{\mathbf{X}}^i] \cdot \dot{\mathbf{h}}^{is} = 0 \qquad (15)$$

It should be noted that Eqs. (12) and (14) are the Lagrange equations corresponding to the quasi coordinates ω and $\underline{\Omega}^a$ and not the Lagrange equations in the corresponding generalized variables. They are also equivalent to the corresponding Roberson-Wittenburg equations written in the vector-dyadic formulation. The equations of translation (13) are equivalent to those obtained by Roberson and the deformation equations presented here can be used even for nonlinear deformations (finite elasticity) and are not restricted to infinitesimal elastic deformations. The system, Eqs. (12–15), suffices to describe the rotational motion of the complete system.

For a satellite on a circular orbit in a gravity field, the gravity terms are, in each equation, similar to the corresponding free rotation terms. Indeed with the previous notations and taking into account the fact that $|\mathbf{R}| \gg |\mathbf{p}^i|$ and $|\rho^i|$, we have

$$d\mathbf{f}^i = -\omega_0^2(\mathbf{R}+\rho^i+\mathbf{p}^i)[1-3\mathbf{R}\cdot(\rho^i+\mathbf{p}^i)/r^2]\,dm$$
$$\mathbf{F}^i = -\omega_0^2 m^i[\mathbf{R}+\rho^i-3(\mathbf{R}\cdot\rho^i)\mathbf{R}/r^2] \tag{16}$$

where r is the norm of $\mathbf{R}$ and ω_0 is the angular velocity of the orbit. The vector $\mathbf{R}$ being directed along the local vertical and $\{\hat{\mathbf{A}}_\alpha\}$ being the orbital frame, with $\hat{\mathbf{a}}_1$ aligned with $\mathbf{R}$, $\hat{\mathbf{a}}_3$ aligned with $\boldsymbol{\omega}_0 = \omega_0\hat{\mathbf{a}}_3$, we will have

$$\mathbf{R}/r = \hat{\mathbf{a}}_1 = [\hat{\mathbf{X}}_\alpha]^T f = [\hat{\mathbf{A}}_\alpha]^T[1 \quad 0 \quad 0]^T$$

The relations (16) allow us to calculate the effect of gravity. The corresponding terms are in Eq. (12)

$$-3\omega_0^2\sum_i\int \mathbf{p}^i \times \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_1 \times \mathbf{p}^i\,dm - 3\sum_{i,j,k} m^j\mathbf{b}^{ij} \times \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_1 \times \mathbf{b}^{kj}$$

in Eq. (14)

$$\omega_0^2\left(3\sum_i T^{bi}\int_i \mathbf{p}^i \times \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_1 \times \mathbf{p}^i\,dm + 3\sum_{i,j,k} T^{bi}m^j\mathbf{b}^{ij} \times \hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_1 \times \mathbf{b}^{kj} + 2\sum_{i,j,k} T^{bi}m^j\mathbf{b}^{ij} \times \mathbf{b}^{kj}\right)$$

in Eq. (13)

$$-\omega_0^2\sum_i(3m_*^{id}\hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_1 \times \mathbf{b}^{id} + 2m_*^{id}\mathbf{b}^{id})$$

and in Eq. (15)

$$-\omega_0^2\left[3\int_i\left(\frac{\partial u^i}{\partial\beta^i}\right)^T[\hat{\mathbf{X}}_\alpha^i]\cdot(\hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_1 \times \mathbf{p}^i)\,dm + 2\int\left(\frac{\partial u^i}{\partial\beta^i}\right)^T[\hat{\mathbf{X}}_\alpha^i]\cdot\mathbf{p}^i\,dm\right]$$

the other terms being implicitly contained in $\mathbf{L}^a$ and $\mathbf{F}^a$.

Motion about an Equilibrium Configuration

The system will be said to be in equilibrium with respect to a nominal rotating frame, $\{\hat{\mathbf{A}}_\alpha\}$, when its reference frame has the same angular velocity $\boldsymbol{\omega}_0$ as the nominal frame and when all the internal variables are constant.

The value of all the vectors and matrices evaluated at this state will be denoted by the subscript (0). The vectors $\mathbf{x}^i$ will be equal to the values of the corresponding $\mathbf{p}^i$ at equilibrium for which all the vectors $\mathbf{u}^i$ are then equal to zero. We will also assume that all the angles describing the relative orientation of the reference frames are defined from the equilibrium configuration. The relative orientation at equilibrium will be kept arbitrary as there could exist obvious physical choices for these frames (alignment with hinge axis, axes of principal moments of inertia,...)

The angles describing relative orientation will always be Tait-Bryan angles in a 1, 2, 3 sequence. For a joint a these angles will be denoted $\underline{\theta}_1{}^a$, $\underline{\theta}_2{}^a$ and $\underline{\theta}_3{}^a$, respectively, the matrix $\underline{\theta}^a$ will then be defined as $\underline{\theta}^a = [\underline{\theta}_1{}^a \quad \underline{\theta}_2{}^a \quad \underline{\theta}_3{}^a]^T$. In order to simplify the analysis, we will also introduce a matrix θ^a related to $\underline{\theta}^a$ by the relation

$$\theta^a = A^{ra}\underline{\theta}^a$$

We will not necessarily define the vectors $\mathbf{z}^a$ from an equilibrium configuration and allow for a nominal value $z_0{}^a$: this will prove interesting, for preliminary design when massless joints are considered. The vector $\mathbf{z}^a$ will then be written as

$$\mathbf{z}^a = \mathbf{z}_0{}^a + \zeta^a \tag{17}$$

When there are less than six degrees of freedom, some of the components of the matrices θ^a and ζ^a may be equal to zero. This will not modify the form of the equations.

The displacements and rotations in the structure will be considered as linear functions of the deformation variables (with $\mathbf{u}^i = 0$ for $\beta^i = 0$).

If $\beta^i = [\beta_1{}^i \ \ldots \ \beta_n{}^i]^T$ is the matrix of deformation variables, the displacements $\mathbf{u}^i$ will be expressed as

$$\mathbf{u}^i = \mathbf{u}^i(\mathbf{x}^i, \beta^i) = [\hat{\mathbf{X}}_\alpha{}^i]^T\underline{W}^{iT}\beta^i = [\hat{\mathbf{X}}_\alpha]^T W^{iT}\beta^i \tag{18}$$

where $W^i = W^i(\mathbf{x}^i)$ and $\underline{W}^i$ are state dependent matrices related by $W^i = A^{ri}\underline{W}^i$. The rotations in the structure will be described by the 3×1 matrices α^i written as

$$\alpha^i = \underline{V}^{iT}\beta^i \tag{19}$$

It will be convenient to define V^i as the matrix related to $\underline{V}^i$ by

$$V^i = A^{ri}\underline{V}^i$$

Using the assumption on the displacements, the first term of Eq. (15) can be written as

$$\int_i(\partial\varepsilon^i/\partial\beta^i)^T\sigma^i\,dV = K^i\beta^i + K_0{}^i$$

where the matrix K^i is symmetrical and $K_0{}^i$ corresponds to the distribution of prestresses in the structure.

The matrices W^i, V^i, K^i, and $K_0{}^i$ must be determined from the structure model (finite elements, modal analysis). The transformation matrices A^{rp} where p indicates a body or a joint can be linearized as functions of Tait-Bryan angles $\phi_\alpha{}^p$ as

$$A^{rp} = (E+\tilde{\phi}^p)A^{rp}$$

where E is a unit matrix and the tilde matrix is a 3×3 skew-symmetric matrix defined from the elements of a 3×1 matrix $v = [v_1 \quad v_2 \quad v_3]^T$ as

$$\tilde{v} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

The ϕ matrices can be related to the abovementioned variables by the relations

$$\phi^i = -\sum_a T^{ai}\theta^a - \sum_{a,j} T^{ai}S^{ja}V_0{}^{ja}\beta^j$$
$$\phi^a = -\sum_{c\leq a}{}' \tau^c\theta^c - \sum_{c\leq a}{}'\sum_j \tau^c S^{jc}V_0{}^{jc}\beta^j + \sum_j S^{ja}V_0{}^{ja}\beta^j \tag{20}$$

where $\tau^c = 1$ when arc c is directed towards r, -1 otherwise; $\sum'_{c\leq a}$ denotes a summation over the indicated indices but with the deletion of the term $c = a$ when $\tau^a = -1$.

The similarity between Eqs. (20) and (9) should be noted. The linearization of the matrix ω in terms of the angles θ_α (describing the orientation of the body frame with respect to $\{\hat{\mathbf{A}}_\alpha\}$) is given by

$$\omega = \omega_0 + \tilde{\omega}_0\theta + \dot{\theta}$$

where $\omega_0 = [0 \quad 0 \quad \omega_0]^T$.

The components of the barycentric vectors given by $\mathbf{b}^{ia(j)} = [\hat{\mathbf{X}}_\alpha]^T b^{ia(j)}$ are from Eqs. (5, 7, 18, and 19) (in linear approximation)

$$b^{ia(j)} = b_0{}^{ia(j)} - \tilde{b}_0{}^{ia(j)}\phi^i - \sum_c S^{ic}(m_*{}^{jc}/m)[(W_0{}^{ic} - \tilde{S}^{ic}\tilde{z}_0{}^c V_0{}^{ic})\beta^i + \tilde{S}^{ic}\zeta^c]$$

Using the following property of tilde matrices

$$(\tilde{a}b)^\sim = \tilde{a}\tilde{b} - \tilde{b}\tilde{a}$$

it proves useful to note that the time derivatives of any vector expressed in the reference frame $\{\hat{\mathbf{X}}_\alpha\}$ are given by

$$\dot{\mathbf{v}} = [\hat{\mathbf{X}}_\alpha]^T(\dot{v} + \tilde{\omega}_0 v - \tilde{v}_0\tilde{\omega}_0\theta - \tilde{v}_0\dot{\theta})$$
$$\ddot{\mathbf{v}} = [\hat{\mathbf{X}}_\alpha]^T[\ddot{v} + 2\tilde{\omega}_0\dot{v} + \tilde{\omega}_0\tilde{\omega}_0 v - \tilde{v}_0\ddot{\theta} - 2\tilde{\omega}_0\tilde{v}_0\dot{\theta} - (2\tilde{\omega}_0\tilde{v}_0\tilde{\omega}_0 - \tilde{v}_0\tilde{\omega}_0\tilde{\omega}_0)\theta]$$

Using the abovementioned variables and matrix properties, we could obtain the linearized matrix equations corresponding to the systems (12–15). Unfortunately, these equations will not provide, in all cases, a system that has the symmetry properties desired for a simple Liapunov stability analysis. Roberson showed that the mass matrix (coefficient of the highest time derivatives) of a system equivalent to Eqs. (12–14), is symmetrical,[7] but the complete symmetry is not obtained when there are prestresses in the joints [and in the structure if Eq. (15) is also considered].

The desired symmetry is obtained by the Lagrange equations of the system expressed in terms of generalized variables. As already mentioned, the system under consideration is equivalent to the Lagrangian system in the generalized variables ζ^a and β^i and in the quasi-coordinates ω and Ω^a but it is quite easy to derive the corresponding Lagrangian system in $\theta\zeta^a\theta^a\beta^i$.

From the variational principle, the term in $\delta\underline{\gamma}^a$ provides

$$\delta\gamma^a\cdot(-\sum T^{ai}\mathbf{L}^i+\mathbf{L}^a)=0$$

where $\mathbf{L}^i$ is defined as

$$\mathbf{L}^i=\int \mathbf{p}^i\times\ddot{\mathbf{p}}^i\,dm+\sum_{j,k}m^j\mathbf{b}^{ij}\times\ddot{\mathbf{b}}^{kj}+\dot{\mathbf{h}}^{is}-\int \mathbf{p}^i\times\mathbf{f}^i\,dm-\sum_j\mathbf{b}^{ij}\times\mathbf{F}^j$$

or

$$\underline{\delta\gamma}^{aT}[\hat{\mathbf{X}}_a{}^a]\cdot(-\sum_i T^{ai}\mathbf{L}^i+\mathbf{L}^a)=0 \tag{21}$$

The virtual change $\delta\underline{\gamma}^a$ can be related to the virtual change of Tait-Bryan angles $\delta\underline{\theta}^a$ by the relation

$$\delta\underline{\gamma}^a=\underline{\Phi}^a\,\delta\underline{\theta}^a$$

where in linear approximation

$$\underline{\Phi}^a=E+\underline{\tilde{\theta}}/2+\bar{U}\underline{\hat{\theta}}^a/2 \tag{22}$$

with

$$\bar{U}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix},\qquad \underline{\hat{\theta}}^a=\begin{bmatrix}0&\underline{\theta}_3&\underline{\theta}_2\\\underline{\theta}_3&0&\underline{\theta}_1\\\underline{\theta}_2&\underline{\theta}_1&0\end{bmatrix}$$

the relative angular velocity being then written as

$$\underline{\Omega}^a=\underline{\Phi}^a\dot{\underline{\theta}}^a$$

It must be noted that this relation is still valid for 1 and 2 degrees-of-freedom joints.

Equation (21) can be written in matrix form as

$$(\delta\underline{\theta}^a)^T\underline{\Phi}^{aT}[-\sum_i A^{ai}T^{ai}\underline{L}^i+\underline{L}^a]=0$$

or

$$(\delta\theta^a)^T[-\sum_i A^{ra}\underline{\Phi}^{aT}A^{ai}T^{ai}\underline{L}^i+A^{ra}\underline{\Phi}^{aT}\underline{L}^a]=0$$

and the Lagrange equation in $\delta\theta^a$ is obtained by equating to zero the terms in brackets.

The generalized joint forces corresponding to a linear restoring mechanism can be written as

$$\underline{\Phi}^{aT}\underline{L}^a=\underline{K}^a\underline{\theta}^a+\underline{L}_0{}^a$$

where K^a is a symmetric matrix and $\underline{L}_0{}^a$ corresponds to the prestresses.

The zeroth-order term (in the variables) must be equal to zero to be around an equilibrium configuration, i.e.,

$$\sum_i T^{ai}L_0{}^i=A_0{}^{ra}\underline{L}_0{}^a\triangleq L_0{}^a \tag{23}$$

If $K^a=A^{ra}\underline{K}^aA^{ar}$ and using Eq. (20)

$$A^{ra}\underline{\Phi}^{aT}\underline{L}^a=K^a\theta^a+L_0{}^a+\sum_{c\leqq a}{}'\tau^c\tilde{L}_0{}^a\theta^c+\sum_{c\leqq a}{}'\sum_j\tau^cS^{jc}\tilde{L}_0{}^aV_0{}^{jc}\beta^j-\sum_j\tilde{S}^{ja}\tilde{L}_0{}^aV_0{}^{ja}\beta^j \tag{24}$$

Using Eqs. (23) and (22)

$$\sum_i A^{ra}\underline{\Phi}^{aT}A^{ai}(-T^{ai}L^i)=\sum_i(-T^{ai}L^i)-\tfrac{1}{2}\tilde{L}_0{}^a\theta^a-\tfrac{1}{2}A_0{}^{ra}(\bar{U}L_0{}^a)^{\wedge}A_0{}^{ar}\theta^a \tag{25}$$

The Lagrange equation in θ^a is obtained by summing Eqs. (24) and (25)

$$\sum_i(-T^{ai}L^i)+L_0{}^a+\sum_{c\leqq a}{}'\tau^c\tilde{L}_0{}^a\theta^c+\sum_{c\leqq a}{}'\tau^cS^{jc}\tilde{L}_0{}^aV_0{}^{jc}\beta^j-\sum_j S^{ja}\tilde{L}_0{}^aV_0{}^{ja}\beta^j+[\tau^a\tilde{L}_0{}^a-\tfrac{1}{2}A_0{}^{ra}(\bar{U}L_0{}^a)^{\wedge}A_0{}^{ar}+K_0{}^a]\theta^a=0 \tag{26}$$

where L^i has to be developed up to first-order terms. It is clear that these equations are equivalent to those obtained by the Hooker procedure[8,9] after projection onto the body reference frame. They could also have been obtained by the procedure adopted in Ref. 2.

The Lagrange equations in the variables θ can be obtained from the equation in the quasi-coordinates ω (coefficients of $\delta\boldsymbol{\psi}$). Indeed if θ are Tait-Bryan angles describing the relative orientation of the body frame with respect to the nominal reference frame, the angular velocity $\boldsymbol{\omega}$ can be written as

$$\boldsymbol{\omega}=\omega_0+[\hat{\mathbf{X}}_\alpha]^T\Phi^0\dot{\theta}=\omega_0+\omega'$$

It can be seen without difficulty[5] that the equations in the quasi-coordinates ω' are also the Euler-Liouville equations of the whole system and as there are no constant generalized forces acting on the system, the corresponding linearized equations are the same as linearized equations in the generalized coordinates θ.

The linearized systems (12, 13, 26, and 15) can be written, as any Lagrangian system about an equilibrium, under the form

$$M\ddot{x}+G\dot{x}+Kx=0$$

where $x=[\theta^T\quad\theta^{a^T}\quad\zeta^{a^T}\quad\beta^{i^T}]^T$.

M and K are symmetric matrices and G is a skew-symmetric matrix.

It should be noted that the sum of constant terms must be equal to zero. This implies that

$$\tilde{\omega}_0I_0\omega_0+\tilde{\omega}_0h_0-\sum_i\tilde{\rho}_0{}^iF_0{}^i-\sum_i\int_i\tilde{x}^if_0{}^i\,dm=0 \tag{27}$$

where I_0 is the total inertia matrix at equilibrium and h_0 is the total external angular momentum at equilibrium

$$-\sum_k T^{ak}\int_k\tilde{x}^k\tilde{\omega}_0\tilde{\omega}_0x^k\,dm-\sum_{i,j,k}T^{ai}m^k\tilde{b}_0{}^{ik}\tilde{\omega}_0\tilde{\omega}_0b_0{}^{jk}-\sum_{k,s}T^{ak}\tilde{\omega}_0h_0{}^{is}+\sum_{i,k}T^{ak}\tilde{b}_0{}^{ki}F_0{}^i+\sum_iT^{ai}\int_i\tilde{x}^if_0{}^i\,dm+L_0{}^a=0 \tag{28}$$

$$\sum_i m_*{}^{ia}(\tilde{\omega}_0\tilde{\omega}_0b_0{}^{ia}+F_0{}^i/m)+F_0{}^a=0 \tag{29}$$

$$K_0{}^i-\int_iW_0{}^{iT}f_0{}^i\,dm+\int_iW_0{}^{iT}\tilde{\omega}_0\tilde{\omega}_0x^i\,dm-\sum_aS^{ia}W_0{}^{iaT}F_0{}^a-\sum_aS^{ia}V_0{}^{iaT}(L_0{}^a+\tilde{S}^{ia}\tilde{z}_0{}^aF_0{}^a)+\sum_sV_0{}^{isT}\tilde{\omega}_0h_0{}^{is}=0 \tag{30}$$

Equation (27) provides the equilibrium conditions for the equivalent rigid gyrostat. For freely spinning gyrostats, they imply that the inertia matrix be equal to

$$I_0=\begin{bmatrix}I_{11}&0&-\alpha J_1\\0&I_{22}&-J_2\\-\alpha J_1&-J_2&I_{33}\end{bmatrix}$$

where $\omega_0J_1=h_{0_1}$, $\omega_0J_2=h_{0_2}$, and

$\alpha=1$ for freely spinning systems

$a=\frac{1}{4}$ for orbiting systems

These conditions are well known. Equations (28) and (29) can, for instance, provide the values of L^a and F^a that are necessary to obtain an equilibrium given the other parameters.

Similarly, Eq. (29) can be used to determine the distribution of prestresses in the system. Otherwise, a set of parameters and/or "large" variables may be considered as unknown and the system (27–30) can be used to compute the values of these unknowns that provide equilibrium. It should be noted that the variables used for equilibrium synthesis may be different from those used in the above equations of motion.

In presence of linear damping that can be associated with the variables $\theta^a\zeta^a\beta^i$ the linearized system can be written as

$$M\ddot{x}+G\dot{x}+Kx=-D\dot{x}$$

where D is a positive definite (symmetric) matrix.

When the damping is complete the asymptotic stability of the linearized system (and also the stability of the corresponding nonlinear system) is ensured when the matrix K is positive definite.

When the eigenvalues of matrix K have negative real parts the corresponding systems are unstable.

Completeness of damping can be verified by the algorithms obtained by Roberson[10] and by veryifying the controllability of the systems by a control of the form $-Du$.[11]

It is nevertheless clear that the damping will not be complete for freely spinning systems in which case one should adopt a procedure similar to the one presented in Ref. 5 and check the positive definiteness of a new matrix K'.

The matrices M, G, K, K', and D can be partitioned, e.g.,

$$K = \begin{bmatrix} K_{11} & K_{12}a & K_{13}c & K_{14}j \\ K_{21}b & K_{22}ba & K_{23}bc & K_{24}bj \\ K_{31}d & K_{32}da & K_{33}dc & K_{34}dj \\ K_{41}i & K_{42}ia & K_{43}ic & K_{44}ij \end{bmatrix}$$

The elements of these matrices can be computed fairly easily and the symmetry of the matrix is checked without too much difficulty when the equilibrium conditions are used. We will not present the developed form in order to save space. For practical applications we developed a computer program that provides the numerical values of the various elements of these matrices.

Nevertheless, we can draw a few general conclusions from the analytical expressions of these matrices. For instance, it can be seen that the condition on K_{11} (or K_{11}' for freely spinning systems) are the same as the stability conditions for an equivalent rigid gyrostat with fictitious damping. For nongyrostatic systems, ($h_0 = 0$), these conditions lead to the maximum axis requirement.

The stability conditions obtained from the positive definiteness of the submatrix

$$K'' = \begin{bmatrix} K_{11} & K_{12}a \\ K_{21}b & K_{22}ba \end{bmatrix}$$

are equivalent to those obtained in Ref. 2 (as could be expected). These conditions enable us to find the minimum joint stiffnesses that ensure stability; the latter are useful for a feasibility study and for the design of the system.

Similarly, the positive definiteness of the submatrix including the terms due to the translational degrees of freedom will permit determination of the minimum translation stiffness of the joints.

As far as stability is concerned, each additional condition is a further restriction on the possible range of system parameters. The equivalent rigid system must then be stable in order to ensure the stability of the deformable systems. The total number of stability conditions is clearly equal to the total number of degrees of freedom. As previously shown, these conditions are very easy to compute and they allow an appropriate adjustment of a set of physical parameters of the system.

Conclusion

We have derived the exact nonlinear rotational equations for a system of interconnected deformable gyrostats in a topological tree configuration. These equations were obtained in vector form, from the Alembert's virtual work principle in a logical and straightforward manner. They generalize previously obtained formalisms and, for systems of interconnected rigid bodies, they are equivalent to those obtained by a proper combination of the Euler equations of the various bodies (Roberson-Wittenburg formalism); they are also the Lagrange equations in translational and deformation generalized variables and in the angular velocities considered as quasi-coordinates. From these equations we obtained, without much difficulty, the Lagrange equations of the system expressed only in terms of generalized variables (the abovementioned generalized variables and Tait-Bryan angles describing the relative orientation between the bodies). These equations are presented in matrix form and are valid for interconnections with one to six degrees of freedom. They appear to be equivalent to the equations obtained by the Hooker procedure and projected onto the system reference frame.

These Lagrange equations were linearized about an equilibrium. For this equilibrium, we allowed for a large nominal translational displacement. This is not strictly necessary, as at equilibrium the system could be considered as point connected, but it can be useful when the mass of the connection is neglected, e.g., for preliminary design.

These linearized equations are suitable for stability analysis and as a by-product we obtained a set of equilibrium conditions which permits at least in principle to perform an equilibrium synthesis. The stability of the system is tested from the positive definiteness of the generalized stiffness matrix of the system (or of a modified stiffness matrix for freely spinning gyrostats). The elements of these matrices can be obtained fairly easily by use of the expression given in the paper and we developed a computer program which provides their numerical value by implementing purely algebraic expressions with the physical parameters of the system. This program also checks the stability and the completeness of the damping.

From the algebraic expression of the stability conditions, we obtained the well-known conditions for particular systems (such as rigid gyrostats and systems of point-connected rigid bodies). In our expressions of these stability conditions, a certain number of parameters, such as stiffness in the joints and stiffness distribution in the structure, may be adjusted to obtain stability for a given configuration.

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